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LATIN CUBES OF ORDER ≤ 5

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If c_n represents the number of reduced Latin cubes of order n then $c_1 = c_2 = c_3 = 1$, $c_4 = 64$ and $c_5 = 40246$. *Moreover if* λ_n and λ'_n denote the number of disjoint isomorphism and isotopy classes of reduced Latin cubes of order n respectively, then $\lambda_n = \lambda'_n = 1$ for $n \leq 3$ while $\lambda_4 = 19$, $\lambda'_4 = 12$, and $\lambda_5 = 1860$.

1. Introduction

A *Latin square* of order n is an $n \times n$ square consisting of n rows and n columns such that each row and each column contains each of the numbers $1, 2, \ldots, n$ exactly once. A Latin square is *reduced* if the first row and first column are in the standard order $1, 2, \ldots, n$. If L_n denotes the number of Latin squares and l_n the number of reduced Latin squares of order n, then clearly $L_n = n!\,(n-1)!\,l_n$. Moreover, if $n \leq 9$ the value of l_n is known (see [1, p. 144]).

2. General theory

A Latin cube $A = (a_{ijk})$ of order n is an $n \times n \times n$ cube consisting of n levels, each containing n rows and n columns such that each of the n^3 elements a_{ijk} is one of the numbers $1, 2, \ldots, n$ and $\{a_{ijk}\}$ ranges over all the numbers $1, 2, \ldots, n$ as one index varies from 1 to n while the other two remain fixed. A Latin cube is *reduced* if the first level is a reduced Latin square and the numbers in the upper left corners of the remaining levels are in the standard order $2, \ldots, n$. If C_n denotes the number of Latin cubes and c_n the number of reduced Latin cubes of order n, then $C_n = n!\,(n-1)!\,(n-1)!\,c_n$.

In a personal communication with one of the authors, Professor E.G. Straus observed that $c_n \geq l_n^2$ so that $C_n \geq L_n^2/n!$. In [2] M. Hall showed that $L_n \geq n!\,(n-1)!\,(n-2)! \cdots (2!)\,(1!)$ and consequently $C_n \geq n!\,[(n-1)!\,(n-2)! \cdots (2!)\,(1!)]^2$.

In order to enumerate the Latin cubes of order n, it suffices to enumerate the reduced Latin cubes of order n. We may reduce the problem even further by considering the concept of isotopic Latin squares.

Two Latin squares of order n are said to be *isotopic* if one can be transformed into the other by rearranging rows, rearranging columns, and renaming elements (see [1, p. 124]). This is clearly an equivalence relation which divides the set of Latin squares of order n into disjoint isotopy classes.

If we are given a Latin square of order n then a certain number of Latin cubes of order n can be generated, each with the property that the first level of the cube is the given square. We may now prove the important

Theorem 1. *If two reduced Latin squares of order n are isotopic then they generate the same number of reduced Latin cubes of order n.*

Proof. Suppose L_1 and L_2 are isotopic reduced Latin squares of order n and suppose L_1 generates u reduced Latin cubes of order n, say $S_1, \ldots, S_u$ each of which has L_1 as its first level. Assume that L_2 generates the reduced Latin cubes $T_1, \ldots, T_v$ of order n each of which has L_2 as its first level. We will first show that $u \leq v$.

Consider a cube S_i whose front face is L_1. Let $L_1^1, L_1^2, \ldots, L_1^n$ denote the n levels of S_i. To each of the n levels perform the sequence of row interchanges, column interchanges and renaming of elements necessary to transform L_1 into L_2 to obtain the squares $M_1^1, M_1^2, \ldots, M_1^n$. Consequently, for each $i = 1, \ldots, n$, M_1^i is a Latin square which is isotopic to L_1^i. Moreover, $L_1^1 = L_1$ and $M_1^1 = L_2$. Rearrange the M's in such a way that the element in the upper left corner of the ith square is i for $i = 1, \ldots, n$. We obtain a sequence $N_1^1, N_1^2, \ldots, N_1^n$. Now form the cube T whose levels are $N_1^1, \ldots, N_1^n$, so that T is reduced and $N_1^1 = L_2$. Hence $T = T_k$ for some $k = 1, \ldots, v$.

Suppose that S_j $(j \neq i)$ is transformed by the above sequence of operations to some T_l. Let $K_1^1, \ldots, K_1^n$ be the n levels of S_j. After applying to $K_1^1, \ldots, K_1^n$ the operations that transform L_1 to L_2, we denote the levels by $C_1^1 = L_2, C_1^2, \ldots, C_1^n$ After rearranging to put the levels in standard order we obtain $D_1^1 = L_2, D_1^2, \ldots, D_1^n$. Suppose $T_l = T_k$ so that $D_1^i = N_1^i$ for $i = 1, \ldots, n$.

Since S_i and S_j are both reduced it follows that $C_1^i = M_1^i$ for $i = 1, \ldots, n$. Since L_2 is isotopic to L_1, we apply to the M's the inverse operations that transform L_2 into L_1. For $i = 1, \ldots, n$ we have $L_1^i = K_1^i$ so that $S_i = S_j$.

Thus if we apply to each S_i $(i = 1, \ldots, u)$ the sequence of row interchanges, column interchanges and renaming of elements necessary to transform L_1 into L_2, we obtain u distinct T's so that $u \leq v$. Similarly we can show that $v \leq u$ so that $u = v$, completing the proof.

Suppose there are m_n isotopy classes $K(n, 1), \ldots, K(n, m_n)$ of reduced Latin squares of order n. The value of m_n is known for $n \leq 8$ (see [1, p. 144]). Let $|K(n, i)|$ denote the number of squares in the class $K(n, i)$.

Corollary. *For each n*

$$c_n = \sum_{i=1}^{m_n} a(n, i)\, |K(n, i)|$$

where $a(n, i)$ is a positive integer.

We now apply the above results to determine the number of Latin cubes of order n for $n \leqslant 5$.

If $n \leqslant 3$ then $c_n = 1$ so that $C_1 = 1$, $C_2 = 2$ and $C_3 = 24$.

3. The case $n = 4$

If $n = 4$ there are two isotopy classes of reduced Latin squares (see [1, p. 144]). The first class contains one square

$$L_1 = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

The second class contains three squares and as a representative we choose the square

$$L_2 = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix}$$

Hence $m_2 = 2$, $|K(4, 1)| = 1$, and $|K(4, 2)| = 3$.

An iterative procedure was used to determine the values of $a(4, 1)$ and $a(4, 2)$. The method of constructing reduced Latin cubes of order 4 consists of starting with say L_1 as the first level of our cube, then sequentially filling in the remaining levels using certain Latin squares of order 4. Of the totality of cubes formed in this manner, we count the number of cubes which are reduced Latin cubes. After performing the original calculations by hand, an IBM370/168 computer was programmed to verify the values so that we may state

Theorem 2.

$$a(4, 1) = 22, \qquad a(4, 2) = 14,$$
$$c_4 = 22(1) + 14(3) = 64,$$
$$C_4 = 4!\,3!\,3!\,64.$$

4. *The case* $n = 5$

If $n = 5$, then $m_5 = 2$, $|K(5, 1)| = 6$, and $|K(5, 2)| = 50$. As representatives of each isotopy class we used the squares

$$L_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{matrix} \quad \text{and} \quad L_2 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \\ 3 & 5 & 1 & 2 & 4 \\ 4 & 3 & 5 & 1 & 2 \\ 5 & 4 & 2 & 3 & 1 \end{matrix}$$

The above mentioned program was then extended to tabulate reduced Latin cubes of order 5. We obtain

Theorem 3.

$$a(5,1) = 941, \qquad a(5,2) = 692,$$
$$c_5 = 941(6) + 692(50) = 40246,$$
$$C_5 = 5!\,4!\,4!\,(40246).$$

5. Classification

Two Latin cubes or order n are *isotopic* if there exists a set of permutations σ_i, $i = 1, 4$ of the numbers $1, 2, \ldots, n$ such that one cube can be transformed into the other by rearranging the rows of each level using σ_1, by rearranging the columns of each level using σ_2, by rearranging the levels using σ_3, and finally by renaming the elements (i.e. the numbers $1, 2, \ldots, n$) of the cube using σ_4. If all four permutations are equal, then the cubes are said to be *isomorphic*. Clearly these are equivalence relations on the set Ω_n of Latin cubes of order n so that Ω_n decomposes into disjoint isotopy classes and each isotopy class decomposes into disjoint isomorphism classes. Let λ_n and λ'_n denote the number of distinct isomorphism and isotopy classes of reduced cubes of order n respectively. For each reduced cube of order n there are $n!\,(n-1)!\,(n-1)! - 1$ non-reduced cubes isotopic to it.

As in [1] two Latin squares of order n are *isomorphic* if one can be transformed into the other by the same rearrangement of rows, columns, and elements. If S_1 and S_2 are reduced isomorphic squares of order n then it is not difficult to show that the isomorphism must leave 1 fixed so that the reduced cubes generated by S_1 pair off isomorphically and hence isotopically, with those generated by S_2.

For $n = 4$, it has been shown (see [1, p. 137]) that the isomorphism and isotopism classes of reduced squares are identical. Therefore to count the distinct isomorphism classes, it is sufficient to classify one collection of 14 cubes and the set of 22 cubes and then to cross-check the results for distinct isomorphism classes. The same procedure is followed in order to count the distinct isotopy classes.

For $n = 5$, it has been shown that one isotopy class of squares (in particular, the one containing squares which generate 692 reduced cubes) contains 5 distinct isomorphism classes while the other isotopy class contains only one isomorphism class. Therefore to count the distinct isomorphism classes, it is sufficient to classify the 5 collections of 692 cubes and the one collection of 941 and then to cross-check the results for distinct isomorphism classes.

With the use of an IBM370/168 computer we are able to construct Table 1.

Table 1

Order of Latin cube	No. of distinct isomorphism classes	No. of distinct isotopy classes	No. of reduced cubes
1	1	1	1
2	1	1	1
3	1	1	1
4	19	12	64
5	1860		40246

Remarks. (1) The number of distinct isotopy classes of reduced Latin cubes of order 5 remains unknown.

(2) In the above cases it was found that for two reduced cubes to be isomorphic their first levels must be isotopic. Whether or not this can be extended to cubes of higher order is unanswered.

(3) For the case of order 4, of the 19 isomorphism classes, 1 contains 1, 15 contain 3, and 3 contain 6 cubes each while of the 12 isotopy classes, 1 contains 1, 7 contain 3, 3 contain 6, and 1 contains 24 cubes.

(4) For the case of order 5, of the 1860 isomorphism classes, 7 contain 2, 12 contain 6, 229 contain 8, 10 contain 12, 30 contain 16, and 1572 contain 24 cubes each.

We now consider several other forms of classification where the isomorphisms and isotopies are defined between the corresponding levels of the cubes.

We say that two Latin cubes are *levelwise isomorphic* (*isotopic*) if the corresponding levels are isomorphic (isotopic) as squares. Moreover, they are *strongly levelwise isomorphic* (*isotopic*) if the corresponding levels are isomorphic (isotopic) as squares under the same isomorphism (isotopy).

Clearly strongly levelwise isomorphic (isotopic) implies levelwise isomorphic (isotopic), but not conversely. Isomorphic (isotopic) does not imply levelwise isomorphic (isotopic) and further, strongly levelwise isotopic does not imply isotopic. If

$$C_1 = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix} \quad \begin{matrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{matrix} \quad \begin{matrix} 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix} \quad \begin{matrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{matrix}$$

and

$$C_2 = \begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1&2&3&4&2&1&4&3&3&4&2&1&4&3&1&2\\
2&1&4&3&1&2&3&4&4&3&1&2&3&4&2&1\\
3&4&2&1&4&3&1&2&2&1&4&3&1&2&3&4\\
4&3&1&2&3&4&2&1&1&2&3&4&2&1&4&3
\end{array}$$

then C_1 is levelwise isomorphic to C_2 but not strongly levelwise isomorphic. If

$$D_1 = \begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1&2&3&4&2&3&4&1&3&4&1&2&4&1&2&3\\
2&3&4&1&1&2&3&4&4&1&2&3&3&4&1&2\\
3&4&1&2&4&1&2&3&2&3&4&1&1&2&3&4\\
4&1&2&3&3&4&1&2&1&2&3&4&2&3&4&1
\end{array}$$

and

$$D_2 = \begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1&2&3&4&2&3&4&1&3&4&1&2&4&1&2&3\\
2&3&4&1&4&1&2&3&1&2&3&4&3&4&1&2\\
3&4&1&2&1&2&3&4&4&1&2&3&2&3&4&1\\
4&1&2&3&3&4&1&2&2&3&4&1&1&2&3&4
\end{array}$$

then D_1 is isomorphic to D_2 but not levelwise isomorphic. Finally if

$$E_1 = \begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1&2&3&4&2&1&4&3&3&4&1&2&4&3&2&1\\
2&1&4&3&1&2&3&4&4&3&2&1&3&4&1&2\\
3&4&2&1&4&3&1&2&1&2&3&4&2&1&4&3\\
4&3&1&2&3&4&2&1&2&1&4&3&1&2&3&4
\end{array}$$

and

$$E_2 = \begin{array}{cccc@{\quad}cccc@{\quad}cccc@{\quad}cccc}
1&2&3&4&2&1&4&3&3&4&1&2&4&3&2&1\\
2&1&4&3&1&2&3&4&4&3&2&1&3&4&1&2\\
3&4&2&1&4&3&1&2&2&1&4&3&1&2&3&4\\
4&3&1&2&3&4&2&1&1&2&3&4&2&1&4&3
\end{array}$$

then E_1 is strongly levelwise isotopic to E_2 but not isotopic.

Theorem 4. *If C_1 and C_2 are reduced Latin cubes of order n which are strongly levelwise isomorphic, then $C_1 = C_2$.*

With the use of an IBM370/168 computer and Theorem 4 we are able to construct Table 2 for the 64 reduced Latin cubes of order 4.

Remark. Of the 62 classes, 61 contain 1 and 1 contains 3 cubes; of the 37 classes, 22 contain 1, 9 contain 2, and 6 contain 4 cubes; and of the 8 classes, 6 contain 4, 1 contains 10, and 1 contains 30 cubes.

Table 2

No. of strongly levelwise isomorphism classes	No. of levelwise isomorphism classes	No. of strongly levelwise isotopy classes	No. of levelwise isotopy classes
64	62	37	8

References

[1] J. Denes and A.D. Keedwell, Latin Squares and Their Applications (Academic Press, New York, 1974).
[2] M. Hall, Distinct representative of subsets, Bull. Amer. Math. Soc. 54 (1948) 922–926.